Ph.D. Qualifying Exam – Spring 2015
Electromagnetics and Energy Systems Area

Table of Useful Taylor Series Expansions for Small $x$:

\[
(1 + x)^n \approx 1 + nx \\
e^x \approx 1 + x \\
a^x \approx 1 + x \ln a \\
\ln(1 + x) \approx x \\
\sin x \approx x \\
\cos x \approx 1 \\
\tan x \approx x
\]

Table of Useful Integrals:

\[
\int \frac{dx}{(a + bx + cx^2)^{3/2}} = \frac{4cx + 2b}{(4ac - b^2)\sqrt{a + bx + cx^2}} + \text{const.}
\]
\[
\int \frac{dx}{x^2 + b^2} = \frac{1}{b} \arctan \left( \frac{x}{b} \right) + \text{const.}
\]
\[
\int \frac{xdx}{x^2 + b^2} = \frac{1}{2} \ln \left( x^2 + b^2 \right) + \text{const.}
\]
\[
\int \frac{dx}{(a + bx + cx^2)^{1/2}} = \frac{1}{\sqrt{c}} \ln \left( 2\sqrt{ac + bcx + c^2x^2} + 2cx + b \right) + \text{const.} \quad [c > 0]
\]
\[
\int a \sin(bx) \, dx = -\frac{a}{b} \cos(bx) + \text{const.}
\]
\[
\int a \cos(bx) \, dx = \frac{a}{b} \sin(bx) + \text{const.}
\]
\[
\int_0^r \frac{a}{x} \sin(bx) \, dx = a \sin(b) \quad \text{Si}(br)
\]
\[
\int_0^r \frac{a}{x} \cos(bx) \, dx = a \cos(b) \quad \text{Ci}(br)
\]
\[
\int ax \sin(bx) \, dx = -\frac{ax}{b} \cos(bx) + \frac{a}{b^2} \sin(bx) + \text{const.}
\]
\[
\int ax \cos(bx) \, dx = \frac{ax}{b} \sin(bx) + \frac{a}{b^2} \cos(bx) + \text{const.}
\]
1. **Transmission Line Matching.** For the transmission line configuration shown below, an engineer wishes to insert a shunt capacitor at a distance \( x \) from the \( C \) – \( D \) junction to match the effective impedance of the combined loads. Assume the characteristic impedances of the lines are \( Z_1 = Z_2 = Z_3 = 50 \, \Omega \), and that the phase velocity, \( v_p \), for each line is equal to the speed of light, \( c \). Smith Charts are provided on the following pages, for your convenience.

(a) (50%) Determine the effective impedance at junction \( C \) – \( D \) that results from the two load resistances, \( Z_{L2} \) and \( Z_{L3} \).

(b) (50%) Determine the distance, \( x \), and the capacitance, \( C \), of the engineer’s shunt capacitor, assuming an operating frequency of 300 kHz. You should express the distance in units of wavelength, \( \lambda \), and the capacitance in Farads.
2. Sphere of charge. The electric field produced by a sphere of charge of radius $a$ with density $\rho(r)$ is given by:

$$E_r = \begin{cases} 
  r^3 + Ar^2 & r \leq a \\
  (a^5 + Aa^4) / r^2 & r \geq a 
\end{cases}$$

(100%) Determine the charge distribution $\rho(r)$ assuming it only varies with $r$. 